On Fuzzy Mapping and Control

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Abstract—**A fuzzy mapping from X to Y is a fuzzy set on $X \times Y$. The concept is extended to fuzzy mappings of fuzzy sets on X to Y, fuzzy function and its inverse, fuzzy parametric functions, fuzzy observation, and control. Set theoretical relations are obtained for fuzzy mappings, fuzzy functions, and fuzzy parametric functions. It is shown that under certain conditions a precise control goal can be attained with fuzzy observation and control as long as the observations become sufficiently precise when the goal is approached.**

I. INTRODUCTION

SINCE its introduction, the concept of fuzziness has been extended to algorithms, learning theory, automata, formal languages, pattern classification, probability theory, and the decision making process [1], [16]. The present extension to fuzzy mapping is pertinent to the study of social and economic systems, some decision making processes, and control systems of incompletely specified processes. In social and economic studies, one often discusses functions of a certain type without explicit expression or even an exact definition. For instance, in economic studies the demand function of a merchandize is defined as the number of units which will be consumed as a function of its price. Implicit in this definition is that the purchasing power of the prospective consumers are known or fixed. However in a closed economy, sale and production of the very merchandize causes a change in associated industries (suppliers, etc.) and affects the purchasing power of the prospective consumers through many chains of interaction. If one neglects these interactions, the demand function is not very useful, or at least it is not the pertinent one for optimum price and production determination. If one includes these interactions, the demand function can only be fuzzily defined. It is no wonder many outstanding economists confine their studies to trends or abstract models and avoid using exact mathematical descriptions.

Going the other extreme, control engineers tend to treat their mathematical models of physical systems as exact and precise though they know that the models are neither. They obtain an optimum solution for the nominal model with possibly an added criterion of minimum sensitivity. However no one can be sure how the system performs if it deviates from the nominal model in some finite way.

It is desirable in both cases to have fuzzy mathematics which represent exactly the inexact state of knowledge. For economics and social studies it may be desirable to have some quantitative measure of the trends so that if there exist two or more opposing trends, the result is not completely unpredictable. For the control of physical systems with uncertainty, it may be desirable to have a "robust" control, robust in the sense that the controller is simple and gives guaranteed performance within the uncertain range. A first step in this direction is the introduction of fuzzy functions. The notion of fuzzy mapping clarifies conceptually what one means by fuzzy function and its inverse. It is consistent with prior work in that the set theoretical relations of union and intersection are treated in the same way as [1]. Nonfuzzy mappings and fuzzy parametric functions are then treated as special classes of fuzzy mappings and functions.

Two related concepts are introduced: fineness and observation. The degree of exactitude in our knowledge is represented by the fineness of the fuzzy set or fuzzy mapping. If our knowledge improves, the fuzzy set representing the state or the fuzzy function representing the system becomes finer. If our knowledge becomes exact, the fuzzy set representing the state becomes a point (membership function equals 1 at the point and 0 elsewhere), and the fuzzy function representing the system becomes an ordinary function. Observation is represented by the observation operator. Since our knowledge on the state is improved by observation, the effect of an observation operator on a set is to make it finer.

In the subsequent sections, the lower case letters represent elements or points in space represented by the corresponding upper case letters. The sign $\wedge$ means taking the minimum of two numbers and the sign $\vee$ means taking the maximum of two numbers. The greek letter μ represents membership function. Other symbols are defined as they are introduced.

II. FUZZY MAPPING AND FUZZY FUNCTION

Definition: A fuzzy mapping f from X to Y is a fuzzy set on $X \times Y$ with membership function $\mu_f(x,y)$. A fuzzy function $f(x)$ is a fuzzy set on Y with membership function

$$\mu_{f(x)}(y) = \mu_f(x,y). \tag{1}$$

Its inverse $f^{-1}(y)$ is a fuzzy set on X with

$$\mu_{f^{-1}(y)}(x) = \mu_f(x,y). \tag{2}$$

A fuzzy function f is said to be finer than a fuzzy function g if

$$\mu_{f(x)}(y) \leq \mu_{g(x)}(y), \qquad \forall x,y \in X,Y. \tag{3}$$

Theorem 1: If f is finer than g, f^{-1} is finer than g^{-1}.

Proof: $\mu_{f^{-1}(y)}(x) = \mu_f(x,y) \leq \mu_g(x,y) \leq \mu_{g^{-1}(y)}(x)$.

III. FUZZY MAPPING OF SET

Let A be a fuzzy set on X. The fuzzy set $f(A)$ is defined as

$$\mu_{f(A)}(y) \equiv \sup_{x \in X} (\mu_A(x) \wedge \mu_f(x,y)). \tag{4}$$

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Theorem 2: Let A, B be two fuzzy sets on X. Then

$$\text{if } \quad A \supseteq B, \qquad f(A) \supseteq f(B) \tag{5}$$

$$f(A \cup B) = f(A) \cup f(B) \tag{6}$$

$$f(A \cap B) \subseteq f(A) \cap f(B) \tag{7}$$

$$\text{if } f \text{ is finer than } g, \qquad \text{then } f(A) \subseteq g(A). \tag{8}$$

Proof of (6): Let $C = A \cup B$

$$\begin{aligned}\mu_{f(C)}(y) &= \sup_{x \in X} (\mu_C(x) \wedge \mu_f(x,y))\\ &= \sup_{x \in X} ((\mu_A(x) \vee \mu_B(x)) \wedge \mu_f(x,y))\\ &= \sup_{x \in X} [(\mu_A(x) \wedge \mu_f(x,y)) \vee (\mu_B(x) \wedge \mu_f(x,y))]\\ &= \sup_{x \in X} (\mu_A(x) \wedge \mu_f(x,y)) \vee \sup_{x \in X} (\mu_B(x) \wedge \mu_f(x,y)).\end{aligned}$$

Proof of (7): Let $D = A \cap B$, then

$$\begin{aligned}\mu_{f(D)}(y) &= \sup_{x \in X} (\mu_D(x) \wedge \mu_f(x,y))\\ &= \sup_{x \in X} (\mu_A(x) \wedge \mu_B(x) \wedge \mu_f(x,y))\\ &\leq \sup_{x \in X} (\mu_A(x) \wedge \mu_f(x,y)) \wedge \sup_{x \in X} (\mu_B(x) \wedge \mu_f(x,y))\\ &= \mu_{f(A)}(y) \wedge \mu_{f(B)}(y).\end{aligned}$$

The proofs of (5) and (8) are trivial and omitted.

IV. FUZZY PARAMETRIC FUNCTION

One special class of fuzzy mapping according to (4) is the *fuzzy parametric function.* Let A denote a fuzzy set on the parametric space R. Let Z denote the product space $X \times R$. The fuzzy set (x,A) on Z is defined by its membership function:

$$\mu_{(x_0,A)}(x,r) = \Delta(x - x_0)\mu_A(r) \tag{9}$$

where the function Δ is defined by

$$\begin{aligned}\Delta(0) &= 1\\ \Delta(x) &= 0, \qquad \forall x \neq 0.\end{aligned} \tag{10}$$

Let f be a fuzzy mapping from Z to Y with its membership function given as

$$\mu_f((x,r),y) = \Delta(y - \psi(x,r)) \tag{11}$$

where ψ is an ordinary mapping from $X \times R$ to Y. According to (4)

$$\begin{aligned}\mu_{f(x_0,A)}(y) &= \sup_{x,r} \{\mu_{(x_0,A)}(x,r) \wedge \mu_f((x,r),y)\}\\ &= \sup_{x,r} \{\Delta(x - x_0)\mu_A(r) \wedge \Delta(y - \psi(x,r))\}\\ &= \sup_{r} \{\mu_A(r) \wedge \Delta(y - \psi(x_0,r))\}.\end{aligned} \tag{12}$$

Equation (12) may be interpreted as a membership function $\mu_A(r)$ on the equality relation

$$y = \psi(x_0,r). \tag{13}$$

The function $\psi(x,r)$ is a fuzzy parametric function with fuzzy parameter r. If, given x_0 and y, (13) is satisfied by more than one value of r, (12) gives the membership of (13) as the largest of $\mu_A(r)$ with r satisfying (13).

The function $f(\ ,A)$ is a fuzzy mapping from X to Y with membership function $\mu_{f(\ ,A)}(x_0,y)$ given by (12). However it has the additional property that derivatives and integrals can be defined:

$$\mu_{f_x(x,A)}(y) \equiv \sup_r \{\mu_A(r)\, \Delta(y - \psi_x(x,r))\} \tag{14}$$

$$\mu_{f^{(-1)}(x,A)}(y) \equiv \sup_r \{\mu_A(r)\, \Delta(y - \psi^{(-1)}(x,r)\} \tag{15}$$

where $f^{(-1)}(x,A) = \int_{x_1}^{x} f(t,A)\, dt$ and $\psi^{(-1)}$ is similarly defined.

The following corollary is a direct result of Theorem 2.

Corollary: Let A, B be two fuzzy sets on R, and $f(\ ,A)$, $f(\ ,B)$ be two fuzzy mappings from X to Y with membership functions given by (12).

If $A \subseteq B$, then $f(\ ,A) \subseteq f(\ ,B)$ and $f(\ ,A)$ is said to be finer than $f(\ ,B)$, and

$$f(\ ,A \cup B) = f(\ ,A) \cup f(\ ,B) \tag{16}$$

$$f(\ ,A \cap B) \subseteq f(\ ,A) \cap f(\ ,B). \tag{17}$$

V. SUPPORT OF FUZZY SET

The support of a fuzzy set A is a nonfuzzy set defined as $S_A = \{x \mid \mu_A(x) \neq 0\}$.

Theorem 3: A and B are fuzzy sets on X. Then

$$\text{if } \quad A \supseteq B, \qquad S_A \supseteq S_B \tag{18}$$

$$S_{A \cup B} = S_A \cup S_B \tag{19}$$

$$S_{A \cap B} = S_A \cap S_B. \tag{20}$$

VI. FUZZY CONTROL SYSTEM

Let $X = E^n$ represent the state space. Let $U \subset E^m$ represent the set of allowed controls. The state of the system is a fuzzy set $p(t)$ on X. The dynamic system is represented by a fuzzy mapping from $X \times U$ into X

$$f\colon \quad X \times U \to X$$

$$\mu_{p(t+1)}(x(t+1)) = \mu_f(x(t),u;\, x(t+1)).$$

The control f, U is said to be *finer* than the control g, V if

$$\forall v \in V, \qquad \exists u \in U \text{ subject to } f(x(t),u) \subseteq g(x(t),v), \quad \text{for all } x(t). \tag{21}$$

VII. OBSERVATION OPERATOR

Definition: An observation on p is a renormalization of a subset of p. Let q be a subset of p and $\bar{q}$ denote the renormalized q. Then:

$$\begin{aligned}\mu_q(x) &< \mu_p(x)\\ \mu_{\bar{q}}(x) &= \frac{\mu_q(x)}{\sup_{x \in X} \mu_q(x)}.\end{aligned} \tag{22}$$

It follows from the definition of S and (22), that $S_A = S_{\bar{A}}$ for any A.

An instrument or means of observation is represented by an operator O. Given O and p, the fuzzy set $\bar{q}$ representing the observed state is not unique. The set of possible $\bar{q}$ given O and p is represented by $O \circ p$. Therefore, by definition,

$$\bar{q} \in O \circ p. \tag{23}$$

After the observation, the state of the system becomes $\bar{q}$. An observation operator O satisfies:

$$O \circ A \subseteq O \circ B, \qquad \text{for all } A \subseteq B. \tag{24}$$

The observation operator is characterized by its indefiniteness. An observation on a fuzzy set p may yield any element from the set $O \circ p$. For every element $\bar{q}$ from the set $O \circ p$, there is a q so that (22) is satisfied. But the converse is not true. A $\bar{q}$ satisfying (22) may not belong to $O \circ p$.

In the subsequent sections, the overhead bar on a symbol representing the observed state will be dropped.

Theorem 4: Let A, B be two fuzzy sets on X such that

$$A \subset O \circ B$$

then

$$S_A \subset S_B. \tag{25}$$

Definition: Let O_1 and O_2 be two observation operators in X, and p be an arbitrary fuzzy set on X. We say that O_2 is *more definite* than O_1 if for each

$$q_2 \in O_2 \circ p, \quad \exists q_1 \in O_1 \circ p \quad \text{subject to} \quad q_2 \subseteq q_1.$$

VIII. FUZZY FEEDBACK CONTROL SYSTEM

A fuzzy feedback control system consists of the following: 1) a fuzzy mapping f such that $f: X \times U \to X$; 2) an observation operator O; 3) a goal set G on X; and 4) a control policy η which maps the observed state to a control u such that $\eta: Q \to U$, where Q is the set of observed fuzzy sets $Q = \{q \mid q \in O \circ p, p = \text{fuzzy set on } X\}$.

The feedback control system works as follows. 1) The initial state is a fuzzy set $p(0)$ on X. 2) An observation is made, and the state of the system becomes $q(0)$ such that $q(0) \in O \circ p(0)$. 3) $u(0) = \eta \circ q(0)$. 4) Since the fuzzy mapping $\mu_f(x(0),u(0);x(1))$ representing the controlled dynamic system is given, $\mu_f(q(0),u(0);x(1))$ is obtained from (4), and $p(1)$ is the fuzzy set with membership function

$$\mu_{p(1)}(x) = \mu_f(q(0),u(0);x). \tag{26}$$

5) $p(1)$ becomes the initial state, and steps 2), 3), 4) are repeated with the time variable increased by 1.

As previously described, the control problem is represented by the set of objects f, O, G, and the feedback control system is represented by f, O, η. The goal G is said to be attainable if there is an η such that $q(t) \subseteq G(t)$, for some t, which may or may not be specified.

Theorem 5: Let $P_1(f_1,O_1,G)$ and $P_2(f_2,O_2,G)$ be two control problems such that f_2 is finer than f_1 and O_2 is more definite than O_1. Then G is attainable in P_2 if it is attainable in P_1.

Remarks: Under the assumption of the theorem, there is policy η_1

$$\eta_1: (O_1 \circ p_1) \to U_1 \tag{27}$$

such that for some t_a

$$q_1(t_a) \subseteq G. \tag{28}$$

The existence of η_2 will be proved by induction. Assume that given $p_2(t)$ there is a possible $p_1(t)$ satisfying

$$p_2(t) \subseteq p_1(t) \tag{29}$$

and $p_1(t)$ is obtained from $p(0)$ by following η_1. It will be shown that there exists a policy η_2 such that for every possible $p_2(t+1)$ as a result of following η_2 there exists a $p_1(t+1)$ which is obtainable by following η_1 and

$$p_2(t+1) \subseteq p_1(t+1) \tag{30}$$

$$q_2(t_a) \subseteq G. \tag{31}$$

Since (29) is valid at $t = 0$, the theorem is then proved.

Lemma 1: Given $q_2(t) \in O_2 \circ p_2(t)$, there is a $q_1(t) \in O_1 \circ p_1(t)$ such that

$$q_2(t) \subseteq q_1(t). \tag{32}$$

Proof: Since O_2 is more definite than O_1 there exists q_1 such that

$$q_1 \in O_1 \circ p_2(t) \tag{33}$$

$$q_2(t) \subseteq q_1. \tag{34}$$

From (24) and (29)

$$O_1 \circ p_2(t) \subseteq O_1 \circ p_1(t)$$

and therefore $q_1 \in O_1 \circ p_1(t)$. Q.E.D.

The control policy η_2 is formulated as follows. Given $\hat{q}_2(t)$, there is a corresponding $\hat{q}_1(t)$ subject to (32) is valid. Using η_1, $\hat{u}_1$ is selected: $\hat{u}_1 = \eta_1 \circ \hat{q}_1(t)$. Since f_2 is finer than f_1, there is a control $\hat{u}_2 \in U_2$ such that (21) $f_2(x(t),\hat{u}_2) \subseteq f_1(x(t),\hat{u}_1)$. This $\hat{u}_2$ is then the selected $\hat{u}_2$

$$\hat{u}_2 \equiv \eta_2 \circ \hat{q}_2(t). \tag{35}$$

It will now be shown that the policy η_2 as formulated gives the desirable results (30) and (31).

From (32) and (5)

$$f_1(q_2(t),\hat{u}_1) \subseteq f_1(q_1(t),\hat{u}_1) = p_1(t+1). \tag{36}$$

From (21)

$$f_2(\hat{q}_2(t),\hat{u}_2) \subseteq f_1(\hat{q}_2(t),\hat{u}_1). \tag{37}$$

Noting that $p_2(t+1) = f_2(\hat{q}_2(t),\hat{u}_2)$, (37) and (36) give (30). Repeating the preceding steps, eventually $p_2(t_a) \subseteq p_1(t_a)$ and Lemma 1 gives $q_2(t_a) \subseteq q_1(t_a) \subseteq G$. Q.E.D.

The preceding theorem shows that if a control goal can be attained with a given control and observation, it can be attained with finer controls and more definite observations. The power of the feedback concept is demonstrated by showing that a precise goal can be attained with rather sloppy control and observation, except that as the goal is approached the observation must be precise. The last required condition is necessary because otherwise one cannot tell whether the goal is attained or not.

IX. EXAMPLE

Consider a one-dimensional control problem as follows. 1) The membership function of the state $p(t)$ is not known, except that its support set is $[\xi_1(t),\xi_2(t)]$. 2) The set of control U has only three members $\{-1,0,1\}$. 3) The fuzzy control f is only partially known:

$$u = -1, \qquad -a < \dot{\xi}_1 < \dot{\xi}_2 < -b < 0 \tag{38}$$

$$u = 0, \qquad \dot{\xi}_1 = \dot{\xi}_2 = 0 \tag{39}$$

$$u = 1, \qquad 0 < b < \dot{\xi}_1 < \dot{\xi}_2 < a. \tag{40}$$

In plain words, the state moves in the selected direction at a speed no lower than b and no higher than a. 4) The goal set is

$$\mu_G(x) = \begin{cases} 1, & g_1 \le x \le g_2 \\ 0, & x < g_1 \text{ or } x > g_2. \end{cases}$$

5) The observation O satisfies the following condition: Let $[K_1,K_2]$ denote the support of q. There exist $m > 0$ such that

$$K_2 - K_1 < g_2 - g_1 - m, \qquad \text{if } q \cap G \ne \phi. \tag{41}$$

The significance of (41) is as follows. It does not matter if the observation is quite inaccurate at a distance, as long as it is sufficiently precise when the goal is approached so that the condition $q \subset G$ can be satisfied with some margin m.

A. Continuous Time Case

The interval τ for applying u can be selected at will before another observation is made. The following control policy is used. 1) If $K_2 < g_2$, and $K_1 < g_1$ choose $u = 1$ for an interval

$$\tau = \frac{g_2 - K_2}{a}. \tag{42}$$

2) If $K_1 > g_1$, and $K_2 > g_2$, choose $u = -1$ for an interval,

$$\frac{K_1 - g_1}{a}.$$

3) If $K_2 < g_2$, and $K_1 > g_1$, choose $u = 0$. Condition 3) represents a system having arrived at its goal.

Assuming that the system is in condition 1) at $t = 0$, the preceding control policy gives:

$$\xi_1(\tau) \ge K_1(0) + b\tau \ge K_1(0) + \frac{bm}{a} \tag{43}$$

$$\xi_2(\tau) \le K_2(0) + a\tau = g_2. \tag{44}$$

Subsequent observation yields

$$K_1(\tau) \ge \xi_1(\tau) \ge K_1(0) + \frac{bm}{a} \tag{45}$$

$$K_2(\tau) \le \xi_2(\tau) \le g_2. \tag{46}$$

As the preceding processes are repeated, K_2 is always less or equal to g_2 and K_1 is increased by at least bm/a at each observation. Therefore 3) is eventually arrived.

B. Discrete Time Case

If the interval τ for each application of u is fixed, the goal is attainable if $\tau \le m/a$. If $\tau > m/a$, there is a possibility of the observed system hunting about its goal set indefinitely.

X. NOISY OBSERVATION

Let q be an observed fuzzy set on X and w be a random vector in x space with zero mean, $\bar{w}^2 = \sigma^2$, and w_k, for $k = 1,2,\cdots$ are independent. A noisy observation q' is a translation of the fuzzy set q by a distance w:

$$\mu_{q'}(x + w) = \mu_q(x). \tag{47}$$

XI. EXAMPLE OF SYSTEM WITH NOISY OBSERVATION

The following modifications are made on the system of Section IX. 1) It has noisy observation O':

$$\mu_q'(x + w) = \mu_q(x).$$

2) The observed state q has a constant support interval

$$K_2 - K_1 = \text{constant} < g_2 - g_1. \tag{48}$$

The support sets of q' and q are related by

$$K_{1,2}' = K_{1,2} + w. \tag{49}$$

Let S and S' denote the actual and observed errors, respectively,

$$S = \tfrac{1}{2}(K_1 + K_2 - g_1 - g_2) \tag{50}$$

$$S' = \tfrac{1}{2}(K_1' + K_2' - g_1 - g_2). \tag{51}$$

Let S_n and S_n' denote the values of S and S' at the nth observation. The proposed control policy is

$$u_n = -\operatorname{sgn} S_n' \tag{52}$$

for an interval

$$\tau_n = a_n |S_n'|. \tag{53}$$

From (38), (40), and (48)–(53):

$$S_{n+1} = -a_n v_n S_n' + S_n \tag{54}$$

where $v_n \in [a,b]$.

Theorem 6: As $n \to \infty$, $S_n \to 0$ in probability, if the following conditions are satisfied by the sequence $\{a_i\}$,

$$a_i > 0, \qquad i = 0,1,2,\cdots. \tag{55}$$

As $n \to \infty$

$$a_n \to 0 \tag{56}$$

$$\sum_{i=0}^{i=n} a_i \to \infty \tag{57}$$

$$\sum_{i=0}^{i=n} (a_i)^2 < C \tag{58}$$

where C is a constant.

Lemma 2: Given any sequence $\{a_i\}$ satisfying (55)–(58), then

$$\prod_{i=m}^{i=\infty} (1 - a_i) = 0, \qquad \text{for any finite } m. \tag{59}$$

Let A_i be defined inductively,

$$A_0 = a_0^2, \qquad A_i = a_i^2 + (1 - a_i)^2 A_{i-1}. \tag{60}$$

Then

$$\lim_{n \to \infty} A_n = 0. \tag{61}$$

Proof (of (61)): From (56), for sufficiently large m,

$$\prod_{i=m}^{i=m'} (1 - a_i)(1 + a_i) = \prod_{i=m}^{i=m'} (1 - a_i^2) < 1. \qquad (62)$$

Since

$$\prod_{i=m}^{i=m'} (1 + a_i) = 1 + \sum_{i=m}^{i=m'} a_i + \text{positive terms}$$

$$> \sum_{i=m}^{i=m'} a_i.$$

From (57) given any $\varepsilon > 0$, $\exists M$ such that for all $m' > M$

$$\prod_{i=m}^{i=m'} (1 + a_i) > \frac{1}{\varepsilon}. \qquad (63)$$

From (62) and (63)

$$\prod_{i=m}^{i=m'} (1 - a_i) < \frac{1}{1/\varepsilon} = \varepsilon. \qquad (64)$$

The limit (59) is proved.

From (58), given any ε, $\exists$ an m such that

$$\sum_{i=m}^{i=\infty} (a_i)^2 < \frac{\varepsilon}{2}.$$

From (59), $\exists$ an M such that

$$\prod_{i=m}^{i=M} (1 - a_i)^2 A_{m-1} < \frac{\varepsilon}{2}.$$

For all $m' > M$

$$A_{m'} = \sum_{k=m}^{k=m'} \prod_{i=k+1}^{i=m'} (1 - a_i)^2 a_k^2 + \prod_{i=m}^{i=m'} (1 - a_i)^2 A_{m-1}$$

$$< \sum_{k=m}^{k=m'} a_k^2 + \frac{\varepsilon}{2} < \varepsilon.$$

Equation (61) is proved.

Proof of Theorem 6: Equation (54) can be written as

$$S_{n+1} = (1 - a_n v_n) S_n - a_n v_n w_n. \qquad (65)$$

Reducing n by 1 and substituting the resulting expression for S_n in (65)

$$S_{n+1} = (1 - a_n v_n)(1 - a_{n-1} v_{n-1}) S_{n-1} - (1 - a_n v_n) a_{n-1} v_{n-1} w_{n-1} - a_n v_n w_n.$$

Repeated substitution gives

$$S_n = S_n' + S_n'' \qquad (66)$$

where S_n' is a fixed component and S_n'' is a random component

$$S_n' = S_0 \prod_{i=0}^{i=n-1} (1 - a_i v_i) \qquad (67)$$

$$S_n'' = -\sum_{k=0}^{k=n-1} \left[\prod_{i=k+1}^{i=n-1} (1 - a_i v_i) \right] a_k v_k w_k. \qquad (68)$$

The expected value of $S_n''^2$ is

$$E[S_n''^2] = \sum_{k=0}^{k=n-1} \left[\prod_{i=k+1}^{i=n-1} (1 - a_i v_i) \right]^2 a_k^2 v_k^2 w^2. \qquad (69)$$

Since $b \leq v_n \leq a$, the conditions (55)–(58) are satisfied by taking $(a_i v_i)$ as a_i. Equation (69) becomes

$$E[S_n''^2] = A_{n+1} w^2. \qquad (70)$$

From (59), (61), (67), and (69):

$$\lim_{n \to \infty} S_n' \to 0, \qquad \lim_{n \to \infty} E[S_n''^2] \to 0.$$

Q.E.D.

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